THE HYDRODYNAMIC FORMULATION OF CERTAIN PROBLEMS IN THE THEORY OF CRACKS

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A simplified formulation is suggested for certain problems concerning the development of cracks in solid bodies under the action of intense pressure. As a model for the solid body an ideal liquid is considered; such a model has already found wide application in the hydrodynamic theory of hollow charges and explosion [1,2] and in the acoustic theory of spalling [3] where extremely high pressures also exist. The proposed simplification provides us with an effective method for solving a number of problems which in a more exact formulation are found to be intractable. The solutions obtained may be of value in themselves within the context of ideal liquids.

1. Hydrodynamic formulation of certain static problems of cracks in solid bodies. 1. Suppose that a solid body, initially at rest, is subjected to extremely large body and surface forces over a certain interval of time Δt. As a model of the solid body, let us consider an ideal liquid. This assumption is evidently justified for high pressures. In addition, let us assume that

$$v \ll c, \quad v\Delta t \ll L \tag{1.1}$$

Here v is the characteristic velocity of particles of the body after the application of the large pressures and body forces, c is the velocity of sound in the body and L is a characteristic linear dimension. If conditions (1.1) are satisfied it can be shown [4] that for the particle velocity $\mathbf{v}$ after impact and the impulse pressures

$$P = \int_0^{\Delta t} p\, dt \qquad (p \text{ is the pressure})$$

there exist the fundamental relations

$$\mathbf{v} = \text{grad } \varphi, \quad \Delta\varphi = 0, \quad \varphi = \mathbf{U} + \rho^{-1}P, \quad \mathbf{F} = \text{grad } \mathbf{U} \tag{1.2}$$

Here $\mathbf{F}$ are the impulse body forces, $\mathbf{U}$ is the potential and Δ is the Laplace operator.

2. In [2,5], which deal with explosion theory, it is assumed that the surfaces formed after the explosion are very smooth. Although such an assumption would be quite justified in the case of noncohesive or slightly cohesive soils such as sand, it is hardly justified for brittle bodies (such as rock masses), since the surfaces formed after an explosion in such bodies are uneven, with cracks branching off into the body. It is therefore of interest to construct a solution still on the basis of the model of an ideal liquid for impact problems in a liquid with cracks.

Suppose that the following condition is satisfied:

$$V\Delta t \gg l \tag{1.3}$$

Here V is the velocity of crack propagation and l is the length of the crack formed after the explosion. Condition (1.3) means that the duration of unsteady crack propagation is small compared with the duration of the action of the intense pressures. Note that the maximum velocity of crack propagation is of the order of the velocity of sound [6,7], and since $v\Delta t \ll l$, equations (1.1) and (1.3) will not be inconsistent.

If condition (1.3) is satisfied, we can evidently assume that the crack develops instantaneously and that the impulse of the cohesive forces during the period of crack propagation is negligibly small compared with their impulse during the time when the crack is stationary. In connection with the latter cohesive forces, we shall adopt the two hypotheses of Barenblatt, namely that the end region of the crack in which the cohesive forces act is small and autonomous [7]. Then the intensity of the cohesive forces will evidently be the same as the intensity of the cohesive forces in a static crack.

Making use of the condition of finiteness of the impulse pressure at the tip of the crack, which is analogous to Khristianovich's condition in the theory of cracks, it is not difficult to obtain the analogy of Barenblatt's condition [7]: the impulse pressure in the region of the tip of the crack has a singularity of the type

$$P = \frac{K\Delta t}{\pi\sqrt{s}} \tag{1.4}$$

where K is the static modulus of cohesion and s is the small distance from the tip of the crack on its extension. Condition (1.4) may be used to determine the previously unknown length of the crack.

Note that when the impulse of the cohesive forces is negligibly small compared with the other forces which resist the development of the crack, one condition of finiteness is sufficient for the determination of the crack length.

Conditions (1.1) are the conditions of statics for a solid body. The displacement u and the pressure p within the body can easily be found from the solution to the problem in its present simple formulation if it is assumed that the pressures acting on the body are independent of time during the interval Δt, and that the velocity $\mathbf{v}$ increases linearly with time during this same interval Δt. Then

$$P = p\Delta t, \qquad \mathbf{u} = {}^1/_2\, \mathbf{v}\Delta t \tag{1.5}$$

and since the problem is linear, the factor Δt cancels in the final formulas. It should be pointed out that we can, of course, solve only those problems in which on the bounding surfaces the normal stresses or normal displacements are given and the tangential stresses are zero.

2. Specific problems. We shall now consider some problems of cracks in solid bodies in the proposed simplified formulation. We shall confine our attention to plane problems, for which the fundamental relations (1.2) may be conveniently written in the form

$$P = \rho \operatorname{Re} f(z), \qquad \mathbf{v} = v_x + iv_y = \overline{f'(z)} \tag{2.1}$$

where $f'(z)$ is an analytic function of $z = x + iy$; v_x and v_y are components of velocity along the axes of Cartesian coordinates x and y. Here it is assumed that impulse body forces are absent ($\mathbf{F} = 0$).

1. *Simple problems*. Consider an infinite body with a cavity of arbitrary shape but of finite dimensions with its surface free from loading. Suppose that an impulse pressure $P = P_\infty$ acts at infinity, as a result of which cracks are propagated from the surface of the body, the cracks also being free from loading. We assume that the crack configuration is known.

It can be shown that in the case when there are one or two cracks the solutions may be written, respectively, in the form

$$f(z) = \frac{P_\infty}{\rho}\sqrt{\frac{g(z) + L}{g(z) - L}}, \qquad f(z) = \frac{P_\infty g(z)}{\rho\sqrt{g^2(z) - L^2}} \tag{2.2}$$

Here $g(z)$ is a function which effects the conformal transformation of the exterior of the contour in the physical plane of z into the exterior of the interval $(-L, +L)$ with correspondence of points at infinity and the ends of sections. It is also not difficult to obtain a solution in general form for any number of cracks which reach the boundary of the body.

As an example, consider the case when the cavity is a circle of radius R with two identical cracks of length l along the x-axis. In this case the transformation function $g(z)$ may be written in the form

$$g(z) = \frac{1}{2}\left(\frac{z}{R} + \frac{R}{z}\right), \qquad L = \frac{1}{2}\left(\frac{l+R}{R} + \frac{R}{l+R}\right) \tag{2.3}$$

and condition (1.4), which determines the length of the crack l, can be reduced to the form

$$\frac{K\Delta t}{\pi P_\infty \sqrt{R}} = \sqrt{\frac{(\lambda+1)(\lambda^2+2\lambda+2)}{2\lambda(2+\lambda)}}, \qquad \lambda = \frac{l}{R} \tag{2.4}$$

The problem of crack interaction is also of interest. The solution for problems with a one-row lattice of cracks, as well as the corresponding results, coincide to the accuracy of a constant multiplier with those obtained in [8, Section 3]. These results will not be repeated here. In a manner analogous to that of [8] we can also study the question of the development of curvilinear cracks.

2. *Explosion in a cylindrical cavity.* Suppose that an explosion of intensity P_0 occurs in a cylindrical cavity of radius R, and that as a result $2n$ symmetrical cracks of the same length l are formed in the body radiating from the boundary of the cavity. At infinity there will be a peak pressure q, so that the corresponding compressive impulse pressure at infinity is $q\Delta t$.

The boundary conditions of the problem can be written in the form

$$\begin{aligned} &P = 0 \quad \text{for} \quad \cot z = k\pi/n \qquad (k = 0, 1, \ldots, 2n;\ |z| < l+R) \\ &P = P_0 \quad \text{for } |z| = R \\ &P = q\Delta t + o(1) \quad \text{as} \quad z \to \infty \end{aligned} \tag{2.5}$$

We rewrite conditions (2.5) in the form of a boundary-value problem for determining the function $f(z)$

$$\begin{aligned} &\mathrm{Re} f(z) = 0 \qquad \text{for } \cot z = k\pi/n \qquad (k = 0, 1, \ldots, 2n;\ |z| < l+R) \\ &\mathrm{Re} f(z) = P_0/\rho \qquad \text{for} \quad |z| = R \\ &f(z) = q\Delta t/\rho + o(1) \qquad \text{as} \quad z \to \infty \end{aligned} \tag{2.6}$$

The solution to (2.6) is of the form [9]

$$f(z)=\frac{P_0}{\pi\rho i\sqrt{\zeta^2-L^2}}\int_{-1}^{+1}\frac{\sqrt{t^2-L^2}}{t-\zeta}\,dt+\frac{q\Delta t\zeta}{\rho\sqrt{\zeta^2-L^2}}= \tag{2.7}$$

$$=\frac{P_0}{\pi\rho i}\left\{\frac{\zeta}{\sqrt{\zeta^2-L^2}}\ln\frac{2\sqrt{1-L^2}+2(1-\zeta)+\zeta^2-L^2}{2\sqrt{1-L^2}+2(-1-\zeta)+\zeta^2-L^2}-\right.$$

$$\left.-\ln\frac{(1+\zeta)\,[\sqrt{(\zeta^2-L^2)(1-L^2)}+\zeta^2-L^2+\zeta(1-\zeta)]}{(1-\zeta)\,[-\sqrt{(\zeta^2-L^2)(1-L^2)}-(\zeta^2-L^2)+\zeta(1+\zeta)]}\right\}+\frac{q\Delta t\zeta}{\rho\sqrt{\zeta^2-L^2}}$$

$$\zeta=\frac{1}{2}\left(\frac{z^n}{R^n}+\frac{R^n}{z^n}\right),\qquad L=\frac{1}{2}\left[\left(1+\frac{l}{R}\right)^n+\left(1+\frac{l}{R}\right)^{-n}\right]$$

$$\sqrt{\zeta^2-L^2}=\zeta+O(\zeta^{-1})\qquad\text{as } \zeta\to\infty$$

We now consider two of the more important particular cases of the general problem. Suppose first of all that $K/q\sqrt{l}\ll 1$, so that the cohesive forces at the tip of the crack are small compared with the forces from the peak pressure. Then the potential $f(z)$ is given by

$$f(z)=-\frac{P_0}{\pi\rho i}\ln\frac{(1+\zeta)\,[\sqrt{(1-L^2)(\zeta^2-L^2)}+\zeta^2-L^2+\zeta(1-\zeta)]}{(1-\zeta)\,[\sqrt{(1-L^2)(\zeta^2-L^2)}-(\zeta^2-L^2)+\zeta(1+\zeta)]} \tag{2.8}$$

The crack length l can be found from the condition of finiteness of impulse pressure at the tip of the crack, which can be reduced to the form

$$\frac{l}{R}=\left(\sqrt{1+\cot^2\frac{\pi q\Delta t}{P_0}}+\cot\frac{\pi q\Delta t}{P_0}\right)^{1/n}-1 \tag{2.9}$$

When $P_0\gg q\Delta t$, condition (2.9) assumes the form

$$\frac{l}{R}=\left(\frac{2P_0}{\pi q\Delta t}\right)^{1/n}-1 \tag{2.10}$$

Suppose now that $K/q\sqrt{l}\gg 1$, so that the cohesive forces at the tip of the crack are large compared with the peak pressure. Then, in expression (2.7) for $f(z)$, we can set $q=0$ and determine the crack length l from condition (1.4), which in this case can be reduced to the form

$$\frac{P_0\sqrt{R}}{K\Delta t}=\left(\frac{n\,[(1+l/R)^n-(1+l/R)^{-n}]}{L\,(1+l/R)}\right)^{1/2}\left[\tan^{-1}\sqrt{\frac{L+1}{L-1}}-\tan^{-1}\sqrt{\frac{L-1}{L+1}}\right]^{-1} \tag{2.11}$$

Expressions (2.11) and (2.7) may be used to find the crack length l. When $P_0\sqrt{R}/K\Delta t\gg 1$, i.e. when the impulse forces are large compared with the cohesive forces acting in the end region of the crack, so that $l\gg R$, formulas (2.11) and (2.7) can be reduced to the very much simplified form

$$\frac{P_0^2R}{K^2(\Delta t)^2} = \frac{n}{2}\left(\frac{l}{R}\right)^{2n-1} \tag{2.12}$$

Formulas (2.9) to (2.12) show that the crack length depends very much on the number of cracks $2n$, which in general is unknown. In order to find this number a further physical condition is required. It is apparent from physical considerations that in the first place n depends on the magnitude of the impulse P_0 and on the properties of the material.

3. An acoustic approximation to some dynamic problems of cracks. For acoustic problems on the motion of an ideal compressible liquid we have the fundamental relations [4]

$$c^2\triangle\varphi = \frac{\partial^2\varphi}{\partial t^2}, \qquad \mathbf{v} = \operatorname{grad}\varphi, \qquad p = -\rho\frac{\partial\varphi}{\partial t} \tag{3.1}$$

Here φ is the potential of the perturbed motion which is characterized by velocity $\mathbf{v}$ and pressure p.

We shall consider two plane problems which illustrate the possibility of using the acoustic model for solving certain dynamic problems in the theory of cracks.

1. *Steady crack propagation.* Suppose that a thin, absolutely rigid semi-infinite wedge moves at a constant velocity V along its axis of symmetry in an infinite body. For simplicity, we shall assume that the thickness of the wedge $2h$ is constant. We assume that the material of the body is an ideal compressible liquid, and that the wedge is preceded by a crack, the length l of which we require to find. Within the framework of an ideal compressible liquid, the cavity formed by the crack may be looked upon as a stagnant zone. We take the x-axis as the axis of symmetry of the wedge with the positive direction in the opposite direction to the motion of the wedge.

Since the thickness of the wedge is assumed to be small the boundary conditions may be specified on the x-axis as follows:

$$\begin{aligned} v_y &= 0 \quad \text{for } x < Vt, \quad x > Vt + l \\ p &= 0 \quad \text{for } Vt < x < Vt + l \end{aligned} \tag{3.2}$$

With the aid of fundamental relations (3.1) for the perturbed velocity and pressure in the steady motion of the liquid we obtain the expressions

$$p = \rho V \operatorname{Re}\Phi'(z), \qquad v_x = \operatorname{Re}\Phi'(z), \qquad v_y = -\sqrt{1-m^2}\operatorname{Im}\Phi'(z)$$
$$z = x - Vt + i\sqrt{1-m^2}y, \qquad m = V/c \qquad (m \leqslant 1) \tag{3.3}$$

Here $\Phi(z)$ is an analytic function. Making use of (3.3), we can write the boundary conditions (3.2) for $\operatorname{Im} z = 0$ in the form

$$\begin{array}{ll} \operatorname{Im} \Phi'(z) = 0 & \text{for } \operatorname{Re} z < 0, \ \operatorname{Re} z > l \\ \operatorname{Re} \Phi'(z) = 0 & \text{for } 0 < \operatorname{Re} z < l \end{array} \tag{3.4}$$

The solution to boundary-value problem (3.4) is of the form [9]

$$\Phi'(z) = \frac{C}{\sqrt{z(z-l)}}, \qquad \sqrt{z(z-l)} = z + O(z^{-1}) \text{ as } z \to \infty \tag{3.5}$$

We determine the real constant C from the obvious condition

$$\int_0^l v_y dx = hV \tag{3.6}$$

and obtain

$$C = \frac{hV}{\pi \sqrt{1-m^2}} \tag{3.7}$$

The crack length l can be found from the condition obtained in [6]

$$l = \frac{\rho^2 c^4 h^2 m^4}{K^2 (1-m^2)} \qquad (K \text{ is the cohesion modulus}) \tag{3.8}$$

The present problem is analogous to the problem of the wedging of brittle bodies [6]. However, the result obtained (3.8) does not agree even qualitatively with the corresponding result in the wedging of brittle bodies [6]. This shows that the analogous and simplified formulation of these problems must be approached with extreme caution. However, (3.8) does provide a reasonable result for an ideal elastic liquid if the crack is looked upon as a stagnant zone.

2. *Nonsteady propagation of cracks*. Consider an infinite elastic ideal liquid subjected to a constant negative pressure $p = -p_0$. Suppose that at the initial instant in time a cavity, which we can represent ideally as a cut along the real axis $(-Vt, Vt)$, where $V \leqslant c$, is initiated at the origin of coordinates and develops along the x-axis at a constant velocity V. The cavity is free from pressure. This problem is analogous to Broberg's problem for elastic bodies [10].

The pressure p satisfies the wave equation

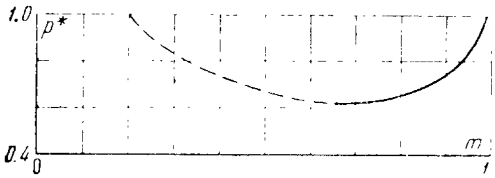
$$c^2 \triangle p = \frac{\partial^2 p}{\partial t^2} \tag{3.9}$$

for which we solve the following boundary-value problem:

$$p = p_0 \quad \text{at} \quad y = 0, |x| < Vt, \qquad p = 0 \quad \text{at} \quad t = 0 \tag{3.10}$$

By superposing a constant pressure $-p_0$ on the solution to this problem we evidently obtain a solution to the initial problem.

The solution to boundary-value problem (3.9) and (3.10) belongs to the class of functional-invariant solutions of Smirnov-Sobolev [11]



$$p = \mathrm{Re}\,\Phi(z) \tag{3.11}$$

$$z = \frac{xt - iy\sqrt{t^2 - c^{-2}(x^2 + y^2)}}{x^2 + y^2}$$

where $\Phi(z)$ is an analytic function.

In the plane of the complex variable z we obtain the following boundary-value problem to determine the function $\Phi(z)$:

$$\begin{aligned} &\mathrm{Re}\,\Phi(z) = p_0 \quad \text{for } \mathrm{Im}\, z = 0,\ |\mathrm{Re}\, z| > 1/V \\ &\mathrm{Re}\,\Phi(z) = 0 \quad \text{for } \mathrm{Im}\, z = 0,\ |\mathrm{Re}\, z| < 1/c \end{aligned} \tag{3.12}$$

After the integrals have been evaluated, the solution to boundary-value problem (3.12) can be reduced to the form

$$\Phi(z) = p_0 - \frac{2p_0 Vz}{\pi i}\sqrt{\frac{z^2 - 1/c^2}{z^2 - 1/V^2}}\left[K(m) + \left(\frac{1}{V^2 z^2} - 1\right)\Pi\left(\frac{\pi}{2}, \frac{1}{c^2 z^2}, m\right)\right] \tag{3.13}$$

$$m = V/c$$

where K and Π are total elliptic integrals of the first and third kinds. The function $\sqrt{(z^2 - c^2)}/\sqrt{(z^2 - V^{-2})}$ is positive on the upper boundary of the cut $(-c^{-1}, c^{-1})$.

In order to determine the rate of widening of the cavity we make use of the condition obtained in [12]. We find that

$$p^* = \frac{p_0\sqrt{c}}{R} = \frac{1}{\sqrt{2}\sqrt{m(1 - m^2)K(m)}} \tag{3.14}$$

where R is the dynamic cohesion modulus [12].

A qualitative representation of expression (3.14) is provided by the diagram in an analogous way to that for the corresponding relation in the elastic problem given in [12]. The problem we have studied here would evidently be of value on its own in the study of cavitation in a liquid under the action of negative pressures.

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